6.2 Dynamics of the Electromagnetic Field

Using the results obtained above, we can show how to rewrite Maxwell's equations in the language of four-vectors. In order to do this, we first of assume that charge is a relativistic scalar, i.e. it is the same in all reference frames. We then introduce a new four-vector, the current density four-vector $\vec{J}$ with contravariant components given by

$$J^{\mu} = \rho_0 u^{\mu}. \tag{6.16}$$

Here the u^{μ} are the contravariant components of the velocity four-vector for which

$$\left.\begin{aligned} u^0 &= \frac{c}{\sqrt{1 - u^2/c^2}} \\ u^1 &= \frac{u_x}{\sqrt{1 - u^2/c^2}} \\ u^2 &= \frac{u_y}{\sqrt{1 - u^2/c^2}} \\ u^3 &= \frac{u_z}{\sqrt{1 - u^2/c^2}} \end{aligned}\right\} \tag{6.17}$$

The current density is evaluated at the point (x, y, z, t) as measured in S, and is determined both by the velocity with which the charges are moving and by the density of charge at this point at this time. However, the charge density ρ_0 is the proper charge density, that is, it is the charge per unit volume as measured in the neighbourhood of the event (x, y, z, t) as measured with respect to a frame of reference that in which the charges at that point are at rest. Thus, in particular, we have

$$J^0 = \rho_0 u^0 = \frac{\rho_0}{\sqrt{1 - u^2/c^2}} c = \rho c \tag{6.18}$$

where ρ given by

$$\rho = \frac{\rho_0}{\sqrt{1 - u^2/c^2}} \tag{6.19}$$

is the charge density in the frame of reference S in which the length of the volume occupied by the charge has been contracted in the direction of motion of the charge as measured in S. To see what this means, we can suppose that we are considering a small volume ΔV_0 which is stationary with respect to the charges within this volume. But these charges are moving with a velocity $\mathbf{u}$ as measured from a frame of reference S. Thus, if we let $\Delta V_0 = \Delta x_0 \Delta y_0 \Delta z_0$, and the charges are moving in the x direction in S, i.e.$u_y = u_z = 0$, then according to the reference frame S, the x dimension of this volume is contracted to a length

$$\Delta x = \sqrt{1 - u^2/c^2} \Delta x_0 \tag{6.20}$$

while the lengths in the other direction are unaffected. Thus, the volume occupied by the charges as measured in S is

$$\Delta V = \Delta x \Delta y \Delta z = \sqrt{1 - u^2/c^2} \Delta x_0 \Delta y_0 \Delta z_0 \tag{6.21}$$

so that

$$\Delta V_0 = \frac{\Delta V}{\sqrt{1 - u^2/c^2}} \tag{6.22}$$

If we let the charge within this volume be ΔQ, then the charge density will be, in S

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{1}{\sqrt{1 - u^2/c^2}} \frac{\Delta Q}{\Delta V_0} = \frac{\rho_0}{\sqrt{1 - u^2/c^2}} \tag{6.23}$$

where we have explicitly used the fact that the charge is the same in both frames of reference, i.e. that charge is a relativistic scalar.

One of the important properties of the current density four-vector follows if we calculate the 'four-divergence' of $\vec{J}$:

$$\vec{\partial} \cdot \vec{J} = \partial_\mu J^\mu = \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} \tag{6.24}$$

$$= \frac{\partial c\rho}{\partial ct} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \tag{6.25}$$

$$= \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \tag{6.26}$$

The last expression expresses the conservation of charge: the term $\nabla \cdot \mathbf{J}$ is the rate at which charge 'diverges' from a point in space, while the time derivative is the rate of change of the charge density at that point. Since charge is conserved, i.e. neither created or destoyed, the sum of these two terms must be zero, i.e.

$$\partial_\mu J^\mu = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{6.27}$$

We can now show that Maxwell's equations can now be written in the form

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \tag{6.28}$$

$$\partial_\nu G^{\mu\nu} = 0. \tag{6.29}$$

where μ_0 is the magnetic permeability of free space. To demonstrate this, it is necessary to merely expand the expressions for the four possible values of the free index in each case. For example, we have, on setting $\mu = 0$ in Eq. (6.28)

$$\partial_\nu F^{0\nu} = \mu_0 J^0 = \mu_0 \rho c \tag{6.30}$$

and, on expanding the left hand side:

$$\partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = \mu_0 \rho c. \tag{6.31}$$

Replacing the partial derivatives by the usual forms in terms of x, y, and z, and noting that $F^{00} = 0$ gives

$$\frac{1}{c}\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_y}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial z} = \mu_0 \rho c. \tag{6.32}$$

Using the fact that $c^2 = (\mu_0 \epsilon_0)^{-1}$ and recognizing that the derivatives on the right hand side merely define the divergence of $\mathbf{E}$, we get

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{6.33}$$

which is Gauss's Law. In a similar way, the other Maxwell's equations can be derived. This is left as an exercise for the reader.